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Values of the polygamma functions at rational arguments

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Abstract

Gauss in 1812, in his celebrated memoir on the hypergeometric series, presented a remarkable formula for the psi (or digamma) function, $\psi(z)$, at rational arguments z, which can be expressed in terms of elementary functions. Davis in 1935 extended Gauss's result to the polygamma functions $\psi^{(n)}(z) (n \in \mathbb{N})$ by using a known series representation of $\psi^{(n)}(z)$ in an elementary yet technical way. Kölbig in 1996, in his CERN technical report, also gave two extensions to $\psi^{(n)}(z)$ by using the series definition of polylogarithm function and the above-known series representation. Here we aim at deriving general formulae expressing $\psi^{(n)}(z)$ $(n \in \mathbb{N}_0)$ as rational arguments in terms of other functions, which will be obtained in two ways. In addition, several special cases are also considered and, as a by-product of our main results, we derive, in a simple and unified manner, all formulae given by Gauss, Davis and Kölbig. Finally, it should be noted that all our results, in view of the relationship between $\psi^{(n)}(z)$ and the Hurwitz zeta function, $\zeta(s, a)$, could be rewritten in the representation of $\zeta(s, a)$.

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1. Introduction and preliminaries

The *polygamma functions* $\psi^{(m)}(z)$ *of order* m, $m \in \mathbb{N}_0$, are defined by (see [1, p 260, equations (6.4.1) and (6.4.10)], [5, p 644, equation (10.(44a))] and [24, p 22, equation (52)])

$$\psi^{(0)} := \psi(z), \qquad \psi^{(n)}(z) := \begin{cases} \dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\psi(z) \\ (-1)^{n+1} n!\zeta(n+1, z) \end{cases}$$

$$(n \in \mathbb{N} := \{1, 2, 3, \ldots\};\ \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}), \tag{1.1}$$

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where $\psi(z)$ and $\zeta(s,a)$ are, respectively, the psi (or digamma) function, given as the logarithmic derivative of the familiar Gamma function $\Gamma(z)$, $\psi(z) = \mathrm{d}\log\Gamma(z)/\mathrm{d}z = \Gamma'(z)/\Gamma(z)$ (see [1, p 258, equation 6.3.(1)], [4, p 13] and [24, p 13, equation (1)]), and the *generalized* (or *Hurwitz*) *zeta function* defined by [24, p 88, equation (1)]

$$\zeta(s,a) := \sum_{k=0}^{\infty}(k+a)^{-s} \qquad (\mathrm{Re}(s) > 1;\ a \notin \mathbb{Z}_0^-). \tag{1.2}$$

Kirchoff was first to apply the polygamma functions in physics, and summation of rational series and evaluation of integrals are some of their classical applications that are still relevant (see, for instance, [8, 15]). They, together with related $\zeta(s,a)$, constantly find new use, and further development of their theory is needed [12]. Recently, for example, the need for summation of series containing $\psi^{(n)}(z)$ has arisen in various fields, Feynman diagram calculations being best known [9, 20].

Gauss [14, pp 33–34 or pp 155–156 in *Werke*] in 1812 proved that $\psi(z)$, for rational arguments, can be expressed in terms of elementary functions as follows (see [24, p 19, equation (47)]):

$$\psi\left(\frac{p}{q}\right) = -\gamma - \frac{\pi}{2}\cot\frac{p\pi}{q} - \log q + \sum_{k=1}^{q-1}\cos\left(\frac{k2\pi p}{q}\right)\log\left(2\sin\frac{k\pi}{q}\right)$$
$$(1 \leqslant p < q;\ p, q \in \mathbb{N}), \tag{1.3}$$

where γ denotes the Euler–Mascheroni constant (see [24, pp 4–6]).

It is noted that most known formulae involving $\psi(z)$ and $\psi^{(n)}(z)$ can be easily derived from their definitions and known formulae, *except for* (1.3). Gauss's proof of (1.3) was later clarified and much simplified by Jensen [16, pp 52–54, or, pp 144–146 (Engl. Transl.)] (cf [4, pp 13–15] and [21, pp 20–23]). There are other proofs of (1.3) (see [6, 17], [13, equation (1.7.3)] or [24, pp 18–19]).

Davis [11] extended the Gauss's theorem to the polygamma function by starting with the series representation of $\psi^{(n)}(z)$ (see equations (1.1) and (1.2)) in an elementary yet a rather technical way. Kölbig [17, pp 4–5, theorem 2 and p 7, theorem 3] presented two formulae for the $\psi^{(n)}(p/q)$ by using the series definition of polylogarithm function in (1.7) and the aforementioned series for $\psi^{(n)}(z)$.

Here we aim at deriving general formulae expressing $\psi^{(n)}(z)$ $(n \in \mathbb{N}_0)$ at rational arguments in terms of other functions, such as the Hurwitz zeta function, Bernoulli polynomials and Clausen functions, which will be obtained in two ways: one is to use an integral representation of $\psi^{(n)}(z)$ (modified the method given in [24, pp 18–19] or [13, equation (1.7.3)]); the other one, inspired by Jensen [16], is to apply Simpson's series multisection formula (see (2.8)) to the polylogarithm function. As a by-product of proving our main results, we also derive, in a simple and unified manner, all formulae given by Gauss, Davis and Kölbig (see remarks 2 and 3).

For our purpose we introduce the following functions. The *Riemann zeta function* $\zeta(s)$ is defined as $\zeta(s) := \zeta(s,1)$. The *Bernoulli polynomials* $B_n(x)$ and *Bernoulli numbers* B_n are, respectively, defined by [1, p 804, equation (23).1.(1)]

$$\frac{t\,\mathrm{e}^{tx}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (|t| < 2\pi) \qquad \text{and} \qquad B_n := B_n(0). \tag{1.4}$$

The *associated* and *generalized Clausen function of order* n, $\mathrm{Gl}_n(\theta)$ and $\mathrm{Cl}_n(\theta)$, are, respectively, defined by [19, p 282, entry A.1(8)]

$$\mathrm{Gl}_n(\theta) := \sum_{k=1}^{\infty} \frac{\cos k\theta}{k^n} \quad (n \text{ is even}) \qquad \text{and} \qquad \mathrm{Gl}_n(\theta) := \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^n} \quad (n \text{ is odd}), \tag{1.5}$$

and [19, p 281, entry A.1(3)]

$$\mathrm{Cl}_n(\theta) := \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^n} \quad (n \text{ is even}) \qquad \text{and} \qquad \mathrm{Cl}_n(\theta) := \sum_{k=1}^{\infty} \frac{\cos k\theta}{k^n} \quad (n \text{ is odd}). \tag{1.6}$$

The *polylogarithm function* $\mathrm{Li}_\nu(z)$ is defined by [19, p 282, entry A.2.7(1)]

$$\mathrm{Li}_\nu(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}$$
$$(\mathrm{Re}(\nu) > 0, |z| \leqslant 1, z \neq 1; \mathrm{Re}(\nu) > 1, |z| \leqslant 1). \tag{1.7}$$

When $n = 1$, the series in (1.7) defines $\mathrm{Li}_1(z) = -\log(1-z)$.

2. Main results and their proofs

We begin by stating the theorem:

Theorem 1. *If p and q are positive integers, then, in terms of the associated Clausen functions $\mathrm{Gl}_n(z)$ and the generalized Clausen functions $\mathrm{Cl}_n(z)$, we have*

$$\left.\begin{matrix}\psi^{(n)}(p/q)\\ \psi^{(n)}(p/q)\end{matrix}\right\} = \pm n! q^n \sum_{s=1}^{q} \left[\mathrm{Gl}_{n+1}(s2\pi/q) \begin{Bmatrix}\cos(s2\pi p/q)\\ \sin(s2\pi p/q)\end{Bmatrix} \right.$$
$$\left. + \mathrm{Cl}_{n+1}(s2\pi/q) \begin{Bmatrix}\sin(s2\pi p/q)\\ \cos(s2\pi p/q)\end{Bmatrix} \right] \qquad (1 \leqslant p \leqslant q) \tag{2.1}$$

and

$$\psi(p/q) = -\gamma - \log q - \sum_{s=1}^{q-1} [\mathrm{Gl}_1(s2\pi/q)\sin(s2\pi p/q)$$
$$+ \mathrm{Cl}_1(s2\pi/q)\cos(s2\pi p/q)] \qquad (1 \leqslant p < q), \tag{2.2}$$

where, for $m \in \mathbb{N}$, the upper $\psi^{(n)}(p/q)$ on the left-hand side of (2.1) corresponds to the case $n = 2m-1$ and the lower $\psi^{(n)}(p/q)$ corresponds to the case $n = 2m$.

Remark 1. We remark that (2.2) is equivalent to the Gauss digamma theorem (1.3). Indeed, this equivalence readily follows since $\mathrm{Cl}_1(\theta) = -\log[2\sin(\theta/2)]$ (see [22, p 726, entry 5.4.2 (10)]) and the sum involving $\mathrm{Gl}_1(\theta) = \theta - 1/2$ (cf [22, p 726, entry 5.4.2 (5)]) equals $-\frac{1}{2}\cot(\pi p/q)$ which can be easily proved by using summation formulae for sine and cosine functions (see [24, p 19, equations (45) and (46)]).

Proof (an integral representation)**.** We recall a known integral representation of $\psi(z)$ (see [24, p 15, equation (13)]):

$$\psi(z) = -\gamma + \int_0^1 \left(1 - t^{z-1}\right)(1-t)^{-1}\,\mathrm{d}t \qquad (\mathrm{Re}(z) > 0), \tag{2.3}$$

which, upon differentiating n times, yields an integral representation of $\psi^{(n)}(z)$:

$$\psi^{(n)}(z) = \int_0^1 \frac{(\log t)^n t^{z-1}}{t-1}\,\mathrm{d}t \qquad (n \in \mathbb{N};\ \mathrm{Re}(z) > 0). \tag{2.4}$$

Taking $z = \frac{p}{q} (1 \leqslant p \leqslant q;\ p, q \in \mathbb{N})$ and $t = x^q$ in (2.4), we obtain

$$\psi^{(n)}\left(\frac{p}{q}\right) = q^{n+1} \int_0^1 \frac{(\log x)^n x^{p-1}}{x^q - 1}\,\mathrm{d}x. \tag{2.5}$$

Letting $\omega := \mathrm{e}^{2\pi\,\mathrm{i}/q} (\mathrm{i} = \sqrt{-1};\ q \in \mathbb{N})$ and decomposing the integrand of (2.5) into a partial fraction, we get

$$\begin{aligned}\psi^{(n)}\left(\frac{p}{q}\right) &= q^n \sum_{\ell=0}^{q-1} \omega^{\ell p} \int_0^1 \frac{(\log x)^n}{x - \omega^\ell}\,\mathrm{d}x \\ &= -q^n \sum_{\ell=0}^{q-1} \omega^{\ell p} \sum_{j=0}^{\infty} \frac{1}{\omega^{\ell(j+1)}} \int_0^1 (\log x)^n x^j\,\mathrm{d}x,\end{aligned}$$

which, upon using the following known result (see [22, p 488, entry 2.6.3(2)]):

$$\int_0^1 x^j (\log x)^k\,\mathrm{d}x = \frac{(-1)^k \Gamma(k+1)}{(j+1)^{k+1}} \qquad (k+1 > 0;\ j+1 > 0)$$

yields

$$\psi^{(n)}\left(\frac{p}{q}\right) = (-1)^{n+1} n! q^n \sum_{\ell=0}^{q-1} \omega^{\ell p} \sum_{j=1}^{\infty} \frac{1}{j^{n+1}\omega^{\ell j}} \qquad (1 \leqslant p \leqslant q;\ p, q, n \in \mathbb{N}). \tag{2.6}$$

Separating (2.6) into real and imaginary parts and considering $\psi^{(n)}\left(\frac{p}{q}\right)$ in (2.6) is real, we obtain

$$\begin{aligned}\psi^{(n)}\left(\frac{p}{q}\right) &= (-1)^{n+1} n! q^n \\ &\quad\times \sum_{\ell=0}^{q-1} \left\{ \cos\left(\frac{2\pi\ell p}{q}\right) \sum_{j=1}^{\infty} \frac{\cos\left(\frac{2\pi\ell}{q} j\right)}{j^{n+1}} + \sin\left(\frac{2\pi\ell p}{q}\right) \sum_{j=1}^{\infty} \frac{\sin\left(\frac{2\pi\ell}{q} j\right)}{j^{n+1}} \right\} \\ &\quad (1 \leqslant p \leqslant q;\ p, q, n \in \mathbb{N}).\end{aligned} \tag{2.7}$$

Finally, setting $n = 2m - 1$ and $n = 2m (m \in \mathbb{N})$ in (2.7), together with (1.5) and (1.6), gives (2.1).

Similarly, starting from (2.3), the formula (1.3) can be obtained and, for details, we refer the reader to [24, pp 18–19]. Thus, from (1.3) by remark 2 we have (2.2). □

Proof (an application of Simpson's series multisection formula)**.** We begin by recalling Simpson's series multisection formula (see, for instance, [23, p 131]):

If $\Phi(x) = \sum_{k=1}^{\infty} a_k x^k$ and let q be fixed, then, for any p

$$q \sum_{k=0}^{\infty} a_{p+qk} x^{p+qk} = \sum_{s=1}^{q} \Phi\left(\omega^s x\right) \omega^{-sp} \qquad \left(1 \leqslant p \leqslant q;\ p, q \in \mathbb{N};\ \omega = \mathrm{e}^{2\pi\,\mathrm{i}/q}\right). \tag{2.8}$$

Now, by making use of Simson's formula and Abel's theorem (see [7, p 148]) we show that the following formulae hold for $\omega = \mathrm{e}^{2\pi\,\mathrm{i}/q}$,

$$\begin{aligned}\psi^{(n)}\left(\frac{p}{q}\right) &= (-1)^{n+1} n! q^n \sum_{s=1}^{q} \left[\cos\left(\frac{s2\pi p}{q}\right) \mathrm{Re}(\,\mathrm{Li}_{n+1}(\omega^s)) \right. \\ &\quad \left. + \sin\left(\frac{s2\pi p}{q}\right) \mathrm{Im}(\,\mathrm{Li}_{n+1}(\omega^s)) \right] \qquad (1 \leqslant p \leqslant q;\ p, q, n \in \mathbb{N})\end{aligned} \tag{2.9}$$

and

$$\psi\left(\frac{p}{q}\right) = -\gamma - \log q - \sum_{s=1}^{q-1}\left[\cos\left(\frac{s2\pi p}{q}\right)\ \mathrm{Re}(\ \mathrm{Li}_1(\omega^s)) + \sin\left(\frac{s2\pi p}{q}\right)\ \mathrm{Im}(\ \mathrm{Li}_1(\omega^s))\right] \qquad (1 \leqslant p < q;\ p, q \in \mathbb{N}). \tag{2.10}$$

In order to prove (2.9) we apply Simson's formula (2.8) to the polylogarithm function (1.7) and obtain

$$\sum_{k=0}^{\infty} \frac{x^{p+qk} q}{(p+qk)^{\nu}} = \sum_{s=1}^{q} \omega^{-sp}\ \mathrm{Li}_{\nu}(\omega^s x) \qquad (\mathrm{Re}(\nu) > 1),$$

which, upon taking the limit as $x \to 1$ and using Abel's theorem, leads to

$$\zeta\left(\nu, \frac{p}{q}\right) = q^{\nu-1} \sum_{s=1}^{q} \omega^{-sp}\ \mathrm{Li}_{\nu}(\omega^s) \qquad (\mathrm{Re}(\nu) > 1), \tag{2.11}$$

with $\zeta(s, a)$ being the Hurwitz zeta function given in (1.2). Here we assume that ν is real with $\nu > 1$. Separating the right-hand side of (2.11) into real and imaginary parts and considering $\zeta(\nu, p/q)$ is real, we get

$$\zeta\left(\nu, \frac{p}{q}\right) = q^{\nu-1} \sum_{s=1}^{q}\left[\cos\left(\frac{s2\pi p}{q}\right)\ \mathrm{Re}(\ \mathrm{Li}_{\nu}(\omega^s)) + \sin\left(\frac{s2\pi p}{q}\right)\ \mathrm{Im}(\ \mathrm{Li}_{\nu}(\omega^s))\right],$$

which, upon setting $\nu = n + 1$ and in view of the relationship between $\psi^{(n)}(z)$ and $\zeta(s, a)$ (1.1), becomes (2.9).

In order to prove (2.10), we recall a well-known formula (see [1, p 259, equation (6.3.16)]):

$$\psi(z) = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{1+k} - \frac{1}{z+k}\right),$$

which, upon setting $z = \frac{p}{q}$ and using Abel's theorem, yields

$$\psi\left(\frac{p}{q}\right) = -\gamma + \lim_{x\to 1-} \sum_{k=0}^{\infty}\left(\frac{x^{p+qk}}{1+k} - \frac{x^{p+qk} q}{p+qk}\right) := -\gamma + \lim_{x\to 1-} S(x). \tag{2.12}$$

If we apply Simson's formula (2.8) to $\mathrm{Li}_1(x) = -\log(1-x)$, we obtain

$$\sum_{k=0}^{\infty} \frac{x^{p+qk} q}{p+qk} = \sum_{s=1}^{q} \omega^{-sp}\ \mathrm{Li}_1(\omega^s x) = -\log(1-x) + \sum_{s=1}^{q-1} \omega^{-sp}\ \mathrm{Li}_1(\omega^s x). \tag{2.13}$$

Next, for $|x| < 1$, we use (2.13) to get

$$\begin{aligned} S(x) &= -x^{p-q} \log(1-x^q) + \log(1-x) - \sum_{s=1}^{q-1} \omega^{-sp}\ \mathrm{Li}_1(\omega^s x) \\ &= -x^{p-q} \log\frac{1-x^q}{1-x} + \left(1 - x^{p-q}\right)\log(1-x) - \sum_{s=1}^{q-1} \omega^{-sp}\ \mathrm{Li}_1(\omega^s x), \end{aligned}$$

and by making use of Abel's theorem, we arrive at

$$\lim_{x\to 1-} S(x) = -\log q - \sum_{s=1}^{q-1} \omega^{-sp}\ \mathrm{Li}_1(\omega^s). \tag{2.14}$$

Now, it follows from (2.12) and (2.14) that

$$\psi\left(\frac{p}{q}\right) = -\gamma - \log q - \sum_{s=1}^{q-1} \omega^{-sp}\,\mathrm{Li}_1(\omega^s),$$

which, similarly as in verifying (2.9), proves (2.10).

Finally, having in mind (1.5) and (1.6) and applying to (2.9), for $n \in \mathbb{N}$, the following relations:

$$\mathrm{Li}_{2n}(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{Gl}_{2n}(\theta) + \mathrm{i}\,\mathrm{Cl}_{2n}(\theta)$$

and

$$\mathrm{Li}_{2n-1}(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{Cl}_{2n-1}(\theta) + \mathrm{i}\,\mathrm{Gl}_{2n-1}(\theta)$$

we prove (2.1). Likewise, the formula (2.2) is obtained by considering (2.10). □

Theorem 2. *If p and q are positive integers, then, in terms of the Bernoulli polynomials $B_n(x)$ and the generalized Clausen functions $\mathrm{Cl}_n(z)$, we have*

$$\left.\begin{matrix}\psi^{(n)}(p/q)\\ \psi^{(n)}(p/q)\end{matrix}\right\} = n!q^n\left[\frac{(-1)^{\lfloor n/2\rfloor}(2\pi)^{n+1}}{2(n+1)!}\sum_{s=1}^{q} B_{n+1}(s/q)\begin{Bmatrix}\cos(s2\pi p/q)\\ \sin(s2\pi p/q)\end{Bmatrix}\right.$$
$$\left.\pm\sum_{s=1}^{q}\mathrm{Cl}_{n+1}(s2\pi/q)\begin{Bmatrix}\sin(s2\pi p/q)\\ \cos(s2\pi p/q)\end{Bmatrix}\right]\qquad\left(\begin{Bmatrix}n=2m-1\\ n=2m\end{Bmatrix};\, m\in\mathbb{N};\, 1\leqslant p\leqslant q\right) \tag{2.15}$$

and

$$\psi(p/q) = -\gamma - \log q + \pi\sum_{s=1}^{q-1} B_1(s/q)\sin(2s\pi sp/q)$$
$$-\sum_{s=1}^{q-1}\mathrm{Cl}_1(2\pi s/q)\cos(2s\pi sp/q)\qquad (1\leqslant p<q). \tag{2.16}$$

Proof. By using the Fourier series of the Bernoulli polynomials $B_n(x)$ (see, e.g., [1, p 805, equation (23.1.16)], [19, p 202, equation (7.60)], [24, p 119, equation (109)]), we get a known formula

$$\mathrm{Gl}_n(2\pi x) = (-1)^{1+\lfloor n/2\rfloor}(2\pi)^n\frac{B_n(x)}{2n!}\qquad(0\leqslant x\leqslant 1;\, n\in\mathbb{N}\setminus\{1\}), \tag{2.17}$$

which also holds true when $n = 1$ if $0 < x < 1$. Note that (2.17) can also be proved by a result of Adamchik [2, equation (9)]. The proof of theorem 2 will now follow from theorem 1, since, respectively, in view of (2.17), the formulae (2.15) and (2.16) follow from (2.1) and (2.2). □

Corollary 1 (Kölbig [17, pp 4–5, theorem 2])**.** *If n, p and q are positive integers, $1\leqslant p<q$, then, in terms of the derivatives of the cotangent function and the generalized Clausen functions $\mathrm{Cl}_n(z)$, we have*

$$\psi^{(2n-1)}(p/q) = -\frac{\pi}{2}\frac{\mathrm{d}^{2n-1}}{\mathrm{d}\theta^{2n-1}}\cot(\pi\theta)\bigg|_{\theta=p/q} + \sum_{s=1}^{q}\sin(s2\pi p/q)\,\mathrm{Cl}_{2n}(2\pi s/q), \tag{2.18}$$

$$\psi^{(2n)}(p/q) = -\frac{\pi}{2}\frac{\mathrm{d}^{2n}}{\mathrm{d}\theta^{2n}}\cot(\pi\theta)\bigg|_{\theta=p/q} - \sum_{s=1}^{q}\cos(s2\pi p/q)\,\mathrm{Cl}_{2n+1}(2\pi s/q). \tag{2.19}$$

Proof. By combining a known formula for $\psi^{(n)}(z)$ (see [24, p 22, equation (55)])

$$\psi^{(n)}(z) - (-1)^n \psi^{(n)}(1-z) = -\pi \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{\cot \pi z\} \qquad (n \in \mathbb{N}_0; 0 < |z| < 1)$$

and $\psi^{(n)}(p/q) \pm \psi^{(n)}(1-p/q)$ that is computed from (2.15), we get the following:

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}\theta^{2n-1}} \cot(\pi\theta)\,\Big|_{x=p/q} = (-1)^n \frac{(2\pi q)^{2n-1}}{n} \sum_{s=1}^{q} B_{2n}(s/q)\cos(s2\pi p/q), \tag{2.20}$$

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}\theta^{2n}} \cot(\pi\theta)\,\Big|_{x=p/q} = (-1)^{n-1} \frac{2(2\pi q)^{2n}}{2n+1} \sum_{s=1}^{q} B_{2n+1}(s/q)\sin(s2\pi p/q),$$
$$(n, p, q \in \mathbb{N}; 1 \leqslant p < q). \tag{2.21}$$

Now, the assertions follow by theorem 2 together with (2.20) and (2.21). □

Corollary 2 [Kölbig [17, p 7, theorem 3]]. *If p and q are positive integers,* $1 \leqslant p < q$*, then, in terms of the Bernoulli numbers* B_n *and the generalized Clausen functions* $\mathrm{Cl}_n(z)$*, we have*

$$\left.\begin{matrix}\psi^{(n)}(p/q)\\ \psi^{(n)}(p/q)\end{matrix}\right\} = \pm n!q^n \Bigg[Q^n_{p,q} \mp (-1)^{\lfloor (n+1)/2\rfloor} \left(\frac{\pi}{q}\right)^{n+1} \sum_{s=1}^{\lfloor (q-1)/2\rfloor} \left\{\begin{matrix}\cos(s2\pi p/q)\\ \sin(s2\pi p/q)\end{matrix}\right\}$$
$$\times \sum_{l=0}^{\lfloor (n+1)/2\rfloor} \frac{(-1)^l q^{2l}}{(n+1-2l)!} \left|2^{2l}-2\right| \frac{|B_{2l}|}{(2l)!} (q-2s)^{n+1-2l}$$
$$+2 \sum_{s=1}^{\lfloor (q-1)/2\rfloor} \mathrm{Cl}_{n+1}(s2\pi/q) \left\{\begin{matrix}\sin(s2\pi p/q)\\ \cos(s2\pi p/q)\end{matrix}\right\}\Bigg] \quad \left(\left\{\begin{matrix}n = 2m-1\\ n = 2m\end{matrix}\right\}; m \in \mathbb{N}\right), \tag{2.22}$$

where

$$Q^n_{p,q} = \left(1 + \tfrac{1}{2}(-1)^p(1+(-1)^q)(2^{-n}-1)\right)\zeta(n+1).$$

Proof. By making use of a well-known formula for $B_n(x)$ [24, p 59, equation (3)], which could be rewritten in the form

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{2k} B_k x^{n-k} = -\frac{1}{2}x^{n-1} + \sum_{k=0}^{\lfloor (n+1)/2\rfloor} \binom{n}{2k} B_{2k} x^{n-2k},$$

since $B_1 = -1/2$ and $B_{2k+1} = 0$, $k \in \mathbb{N}$, we, by theorem 2, after much tedious algebra, obtain the Kölbig formulae in (2.22). □

Remark 2. In view of (1.1) it is clear that all the above results could be rewritten in the representation of the $\zeta(s,a)$. Moreover, we also establish a new identity for $\psi^{(n)}(z)$, given by theorem 3, which, for any fixed n and q, involves all q values of $\psi^{(n)}(p/q)$ $(p = 1, \ldots, q)$.

Remark 3. Observe that the both Kölbig extensions of the Gauss digamma theorem, given, respectively, by (2.18) together with (2.19) and by (2.22), follow without difficulty from theorem 2. It should be however noted that the first assertion of theorem 2, which we have been unable to find in the literature, resembles the formula proved by Kölbig (see corollary 2). Obviously, our formula (2.15) is much simpler and more compact than (2.22) and involves the Bernoulli polynomials instead of the Bernoulli numbers. Further, note that Davis [11] deduced two separate formulae for $\psi^{(n)}(p/q) \pm \psi^{(n)}(1-p/q)$ which, upon adding, yields (2.7).

Theorem 3. *If p and q are positive integers,* $1 \leqslant p \leqslant q$*, then, in terms of the Bernoulli polynomials* $B_n(x)$ *and the generalized zeta function* $\zeta(s, a)$*, the following identity holds:*

$$\left.\begin{matrix}\psi^{(n)}(p/q)\\ \psi^{(n)}(p/q)\end{matrix}\right\} = n!q^n \sum_{s=1}^{q} \left[\frac{(-1)^{\lfloor n/2 \rfloor}(2\pi)^{n+1}}{2(n+1)!} B_{n+1}(s/q) \begin{Bmatrix}\cos(s2\pi p/q)\\ \sin(s2\pi p/q)\end{Bmatrix} \right.$$
$$\left. \pm \frac{1}{q^{n+1}} \left(\sum_{r=1}^{q} \zeta(n+1, r/q) \begin{Bmatrix}\sin(r2\pi s/q)\\ \cos(r2\pi s/q)\end{Bmatrix} \right) \cdot \begin{Bmatrix}\sin(s2\pi p/q)\\ \cos(s2\pi p/q)\end{Bmatrix} \right], \tag{2.23}$$

where, for $m \in \mathbb{N}$*, the upper* $\psi^{(n)}(p/q)$ *on the left-hand side of (2.23) corresponds to the case* $n = 2m - 1$ *and the lower* $\psi^{(n)}(p/q)$ *corresponds to the case* $n = 2m$*.*

Proof. Cvijović and Klinowski [10, equation (10a)] proved formulae which can be specialized in the following form:

$$\mathrm{Cl}_n\left(\frac{2\pi p}{q}\right) = \frac{1}{q^n} \sum_{s=1}^{q} \zeta\left(n, \frac{s}{q}\right) \begin{Bmatrix}\cos(s2\pi p/q)\\ \sin(s2\pi p/q)\end{Bmatrix} \qquad \left(\begin{Bmatrix}n = 2m+1\\ n = 2m\end{Bmatrix}\right)$$
$$(n, q \in \mathbb{N};\ p \in \mathbb{Z} := \mathbb{N} \cup \mathbb{Z}_0^-). \tag{2.24}$$

Evidently, by theorem 2 and (2.24) we have (2.23). □

3. Special cases

In conclusion, in order to demonstrate an application of the presented results, some special cases of our formulae for the polygamma functions at rational arguments are recorded here (cf [3] and [18]). Since the values of $\psi^{(n)}(p/q)$ are expressed in terms of the Clausen functions, Bernoulli polynomials and Hurwitz zeta function, it is natural to know some of their properties such as

$$B_{2n}\left(\frac{1}{2}\right) = \left(2^{1-2n} - 1\right) B_{2n} \qquad (n \in \mathbb{N});$$

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \qquad (n \in \mathbb{N}_0);$$

and

$$\zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \qquad (m \in \mathbb{N} \setminus \{1\}).$$

We also need the relations

$$\psi^{(2n)}\left(\frac{1}{6}\right) = 2\psi^{(2n)}\left(\frac{1}{3}\right) \qquad (n \in \mathbb{N})$$

and

$$\psi^{(2n)}\left(\frac{5}{6}\right) = -2\psi^{(2n)}\left(\frac{1}{3}\right) - \psi^{(2n)}\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}),$$

which are easily obtained by recalling the multiplication formula for $\psi^{(n)}(z)$:

$$\psi^{(n)}(mz) = \sum_{j=1}^{m} \psi^{(n)}\left(z + \frac{j-1}{m}\right) \qquad (n, m \in \mathbb{N}).$$

Now, we give the value of the simple case of (2.15) when $p = 1$ and $q = 2$:

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! \left(2^{n+1} - 1\right) \zeta(n+1) \qquad (n \in \mathbb{N}), \tag{3.1}$$

which is a well-known formula given, for instance, in [1, p 260, equation (6.4.(4))]. Similarly, we have

$$\psi^{(2n)}\left(\frac{1}{3}\right) = -\psi^{(2n)}\left(\frac{2}{3}\right) = \frac{(2n)!}{2}(1-3^{2n+1})\zeta(2n+1) + (-1)^n \frac{\sqrt{3}(2\pi)^{2n+1}}{6(2n+1)}\left\{-\frac{6n+1}{2} + \sum_{j=1}^{n}\binom{2n+1}{2j}B_{2j}3^{2j}\right\} \qquad (n \in \mathbb{N}), \tag{3.2}$$

and

$$\psi^{(2n)}\left(\frac{1}{4}\right) = \psi^{(2n)}\left(\frac{1}{2}\right) - \psi^{(2n)}\left(\frac{3}{4}\right) = (2n)!2^{2n}(1-2^{2n+1})\zeta(2n+1) + (-1)^{n+1}\frac{(4n+1)(2\pi)^{2n+1}}{4(2n+1)} + (-1)^n \frac{(2\pi)^{2n+1}}{4(2n+1)}\sum_{j=1}^{n}\binom{2n+1}{2j}B_{2j}4^{2j} \qquad (n \in \mathbb{N}). \tag{3.3}$$

Finally, several further special values are

$$\psi^{(2n-1)}\left(\frac{1}{4}\right) = (2n-1)!2^{4n-1}\beta(2n) + (-1)^{n-1}2^{2n-2}(2^{2n}-1)B_{2n}\frac{(2\pi)^{2n}}{2n} \tag{3.4}$$

$$\psi^{(2n-1)}\left(\frac{3}{4}\right) = -(2n-1)!2^{4n-1}\beta(2n) + (-1)^{n-1}2^{2n-2}(2^{2n}-1)B_{2n}\frac{(2\pi)^{2n}}{2n}, \tag{3.5}$$

where $\beta(s) := (\zeta(s, 1/4) - \zeta(s, 1/4))/4^s$.

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